Complex function $f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots$

if $|z| < 1$.

$1 + \sum_{n=1}^{\infty} z^n$ is a series called the geometric series.

more general $\quad a, z \in \mathbb{C}$.

$$f(z) = a + \sum_{n=1}^{\infty} a \cdot z^n.$$

i.e. $f(z) = \lim_{N \to \infty} S_N$, where

$$S_N = a + \sum_{n=1}^{N} a z^n$$

Observe

$$S_N = a + \cancel{az} + \cancel{az^2} + \cdots + \cancel{az^N}$$

$$z S_N = \cancel{az} + \cancel{az^2} + \cdots + \cancel{az^N} + az^{N+1}$$

subtract:

$$(1-z) S_N = a - az^{N+1}$$

$$\Rightarrow S_N = \frac{a - az^{N+1}}{1-z} = a\frac{(1-z^{N+1})}{1-z}$$

$$S_N = \sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}, \quad z \neq 1$$

$$\lim_{N\to\infty} S_N = \frac{1 - \lim_{N\to\infty} z^{N+1}}{1-z}$$ by alg. limit thm

$|z|<1$ exists $\iff |z|<1$.

In this case $\lim_{N\to\infty} S_N = \frac{1}{1-z}$

$$\therefore \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ for } |z|<1.$$

(Taylor series of $\frac{1}{1-z}$ at $z=0$)

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also $\sum_{n=0}^{\infty} a\cdot z^n = \frac{a}{1-z}$ for $|z|<1$.

Other important examples of holomorphic fcns:

$$f(z) = \frac{az+b}{cz+d}$$ for a,b,c,d constants in $\mathbb{C}$ such that $ad-bc\neq 0$.

Associated to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det\neq 0$, you get such a fcn.

These are called "linear fractional transformations" (LFTs).

Cool facts about these:

① Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ s.t. $ad-bc \neq 0$, $eh-fg \neq 0$

then $f(z) = \dfrac{az+b}{cz+d}$ $\quad g(z) = \dfrac{ez+f}{gz+h}$

Then $(f \circ g)(z) = f(g(z))$ is another linear fractional transformation, whose matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Start of proof: $(f \circ g)(z) = \dfrac{a\left(\frac{ez+f}{gz+h}\right)+b}{c\left(\frac{ez+f}{gz+h}\right)+d}$

$$= \underset{\text{(bunch of algebra)}}{\cdots} = \frac{(ae+bg)z + (af+bh)}{\cdots z + \cdots}$$

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② Holomorphic as maps from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$

1-1 and onto.

③ LFTs map (lines & circles) to (lines & circles).

or   or

ie (circles on Riemann sphere) to (circles on the Riemann sphere).

④ eg. $f(z) = \frac{1}{z} = \frac{0z+1}{1z+0}$

$f(z) = z = \frac{1z+0}{0z+1}$.

more general. $f(z) = \frac{z+1}{iz-2}$

Curves in $\mathbb{C}$ and integrals over curves in $\mathbb{C}$.

Smooth curve ← (Ken) C^∞ curve

$\alpha : I \to \mathbb{C}$ s.t. $\alpha'(t) \neq 0 \ \forall t \in I$

some interval in $\mathbb{R}$ (closed or open)

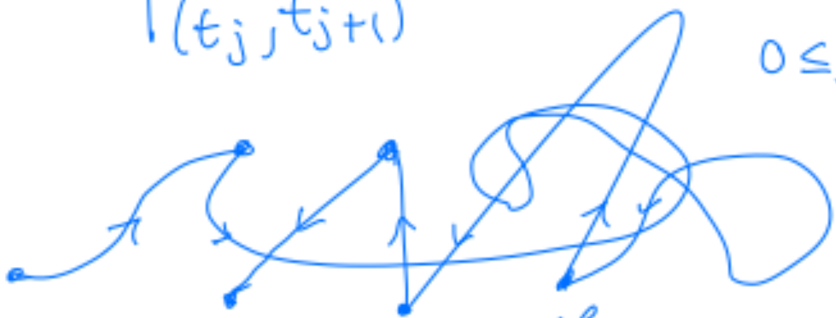
Some books: smooth $= C^1$ and $\alpha'(t) \neq 0 \ \forall t$.

piecewise smooth curve:

$$\alpha : [t_0, t_k] \to \mathbb{C} \quad \text{continuous}$$

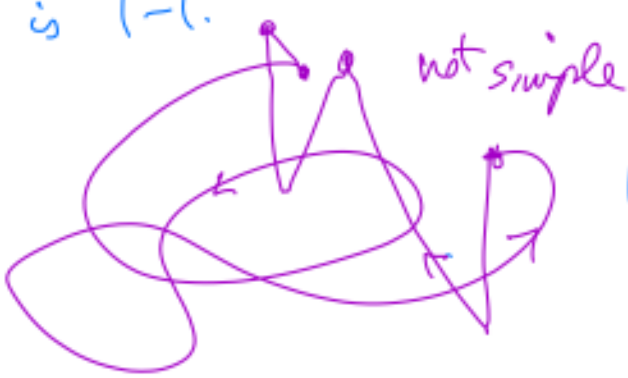
$$t_0 < t_1 < t_2 < \dots < t_k$$

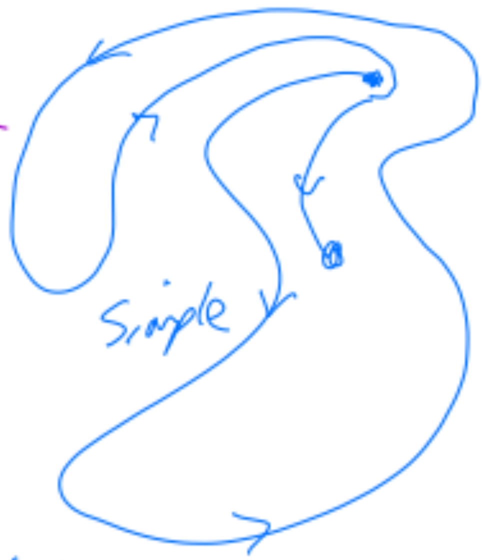
and $\alpha\Big|_{[t_j, t_{j+1}]}$ is smooth. $\forall j$ s.t. $0 \leq j \leq k-1$.



piecewise smooth.

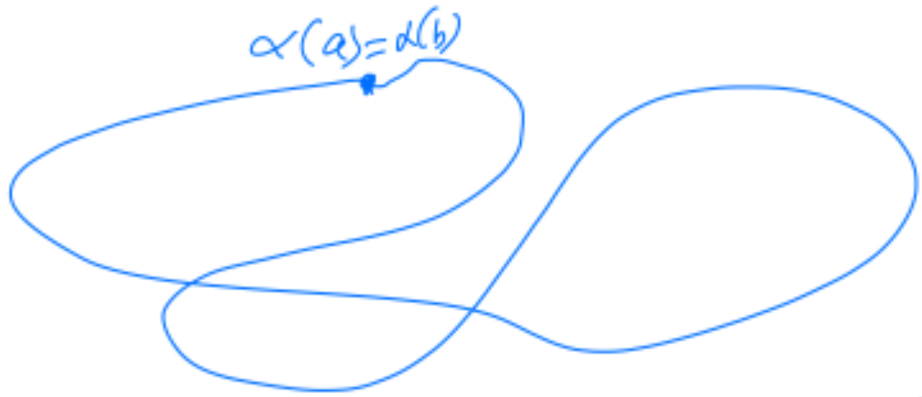
A piecewise smooth curve or contour. is called simple if it is 1-1.



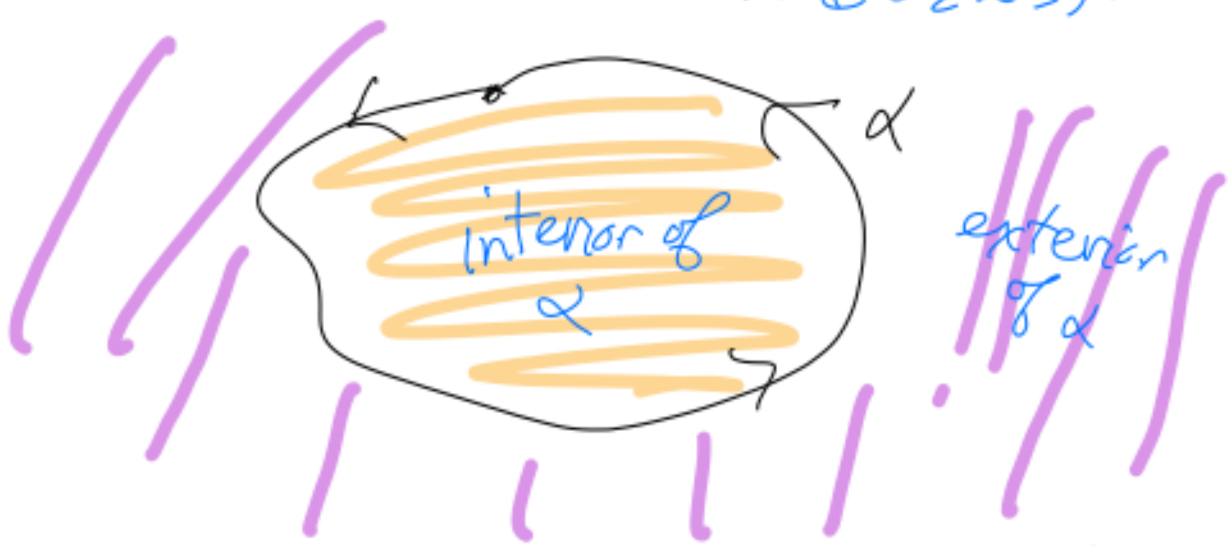


i.e. simple = not self-intersecting.

A curve $\alpha : I \to \mathbb{C}$ is called closed if $I = [a,b]$ and $\alpha(a) = \alpha(b)$.



Let α be a simple, piecewise smooth, closed curve. By Jordan Curve Theorem, it divides $\mathbb{C}$ into two open sets, the <u>interior</u> of α and <u>exterior</u> of α. (exterior — part closest to ∞ in $\mathbb{C} \cup \{\infty\}$).



We call α "<u>positively oriented</u>" if it goes CCW (Counterclockwise), ie. if the interior of α is on your left as you move in the direction of the curve.

(The curve above is positively oriented)

Otherwise, the curve is negatively oriented.

The length of a piecewise smooth curve α, if it exists, for $\alpha : [a,b] \to \mathbb{C}$, is defined to be

$$\int_a^b |\alpha'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$\sqrt{\Delta x^2 + \Delta y^2} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t}\Delta t = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$$\text{length} = \lim_{\substack{\Delta t \to 0 \\ \left(\substack{\Delta x \to 0 \\ \Delta y \to 0}\right)}} \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t \to \int |\alpha'(t)|\,dt$$

You can actually still define the length of a non-differentiable curve by:-

$$\lim_{n \to \infty} \sum_{k=1}^{n} s_k$$

Complex integrals = line integrals in $\mathbb{R}^2$.

Recall

$$\oint_\alpha A(x,y)\,dx + B(x,y)\,dy$$

, where $\alpha : [a,b] \to \mathbb{R}^2 = \mathbb{C}$ smooth

$\alpha(t) = (x(t), y(t))$

$\alpha'(t) = (x'(t), y'(t))$ velocity vector.

$dx = x'(t)\,dt$

$dy = y'(t)\,dt$

$$\int_a^b A(x(t), y(t))\,x'(t)\,dt + B(x(t),(y(t)))\,y'(t)\,dt$$

1-variable integral.

Complex Version

$$dz = dx + i\,dy$$

$f(z)$ complex fcn.

$f : U_{\text{open}} \to \mathbb{C}$, $U \subseteq \mathbb{C}$.

$$\int_\alpha f(z)\,dz$$

$\alpha : [a,b] \to \mathbb{C}$

$\alpha(t) = x(t) + i\,y(t)$ $\quad t \in \mathbb{R}$

$\alpha'(t) = x'(t) + i\,y'(t)$.

$$\int_{\alpha} f(z)\,dz = \int_a^b \underbrace{f\big(x(t)+i\,y(t)\big)\overbrace{\big(x'(t)\,dt + i\,y'(t)\,dt\big)}^{(dx\,+\,i\,dy)}}$$

$$= \int_a^b \mathrm{Re}\big(\underline{\qquad\qquad}\big)\,dt + i\int_a^b \mathrm{Im}\big(\underline{\qquad\qquad}\big)\,dt$$